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LETTER TO THE EDITOR

Exact critical surface of the s-state Potts model with anisotropic interactions on the triangular and honeycomb lattices

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Abstract. Exact expressions for the critical surfaces of paramagnetic–ferromagnetic transitions for the s-state Potts model with anisotropic nearest-neighbour interactions on the triangular and honeycomb lattices are obtained. The derivation, which is based on the duality and star–triangle transformations and a uniqueness argument, is a generalisation of that given by Kim and Joseph for the isotropic case.

In this Letter the s-state Potts model (Potts 1952) with anisotropic nearest-neighbour interactions on the triangular and honeycomb lattices is considered. The model has the Hamiltonian

$$-\mathscr{H}/k_{\mathrm{B}}T = \sum_{\langle ij\rangle} K_{ij}\delta_{\sigma_i\sigma_j} \tag{1}$$

where the δ is a Kronecker delta and the variables σ_i take the values $1, 2, \ldots, s$. By anisotropic interactions we mean that couplings K_{ij} have different values K_1, K_2, K_3 for the three distinct bond directions of the two lattices. For the case of isotropic ferromagnetic couplings Kim and Joseph (1974) have obtained exact expressions for the critical temperatures for the two lattices. Their derivation is based on the duality and star–triangle transformations and a uniqueness argument. In an earlier paper Stephen and Mittag (1972) pointed out that the star–triangle transformation can be applied at the duality point of the Potts model but did not give explicit formulae for the critical temperatures.

We have generalised the derivation of Kim and Joseph to include anisotropic interactions and have obtained exact expressions for the critical surfaces of paramagnetic–ferromagnetic transitions for the two lattices in the variables K_1, K_2, K_3. These expressions could prove useful in applying the differential real-space renormalisation-group equation of Hilhorst *et al* (1978), which has reproduced exact results for the Ising model, to the Potts model. This renormalisation equation generates anisotropic couplings, even if the initial couplings are isotropic.

The derivation begins with the application of the standard duality transformation (Potts 1952, Kihara *et al* 1954) to the Potts model on the triangular lattice with couplings K_α, $\alpha = 1, 2, 3$, to obtain an equivalent Potts model on the honeycomb lattice with couplings $\tilde{K}_\alpha$. In terms of the variables $x_\alpha = \exp(K_\alpha) - 1$, $\tilde{x}_\alpha = \exp(\tilde{K}_\alpha) - 1$ the

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duality transformation has the form

$$\tilde{x}_\alpha = s/x_\alpha. \tag{2}$$

Next the star–triangle transformation (Wannier 1945, Syozi 1972, Kim and Joseph 1974) is applied. It converts the honeycomb lattice to a triangular lattice with new nearest-neighbour couplings K'_α and three-spin couplings of the form $L'\delta_{\sigma_1\sigma_2}\delta_{\sigma_2\sigma_3}\delta_{\sigma_1\sigma_3}$ for each elementary triangle of the lattice. Combining both transformations and introducing $x'_\alpha = \exp(K'_\alpha)-1$, $y' = \exp(L')-1$, one finds

$$x'_\alpha = sx_\alpha/D \tag{3}$$

$$y' = \frac{s^2(D-s)x_1x_2x_3}{(S+sx_1)(S+sx_2)(D+sx_3)} \tag{4}$$

where

$$D(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_1x_3 + x_1x_2x_3. \tag{5}$$

The transformation equations (3)–(5) have a fixed surface ($x'_\alpha = x_\alpha$, $y' = y = 0$) with vanishing three-spin interaction for all x_α satisfying

$$D(x_1, x_2, x_3) = s. \tag{6}$$

The fixed surface for $s = 3$ is shown in figure 1(a). It divides the space of physical couplings $-1 \leqslant x_\alpha \leqslant \infty$ into the two domains $D < s$ and $D > s$.

We now argue on the basis of a uniqueness assumption that the fixed surface and the critical surface coincide. Note that the mapping defined by (3) preserves the ratios x_α/x_β, i.e. the original point and its image lie on the same ray through the origin in x space. For simplicity we consider first of all the case of ferromagnetic couplings $x_\alpha > 0$. The mapping for a typical ray which intersects a point on the critical surface with all x_α positive is shown schematically in figure 2(a). The radial coordinate $r' = (x_1'^2 + x_2'^2 + x_3'^2)^{1/2}$ specifies the distance from the origin of a point on the ray. r^* denotes the fixed point at which the ray intersects the fixed surface $D = s$. The segment $y = 0$,

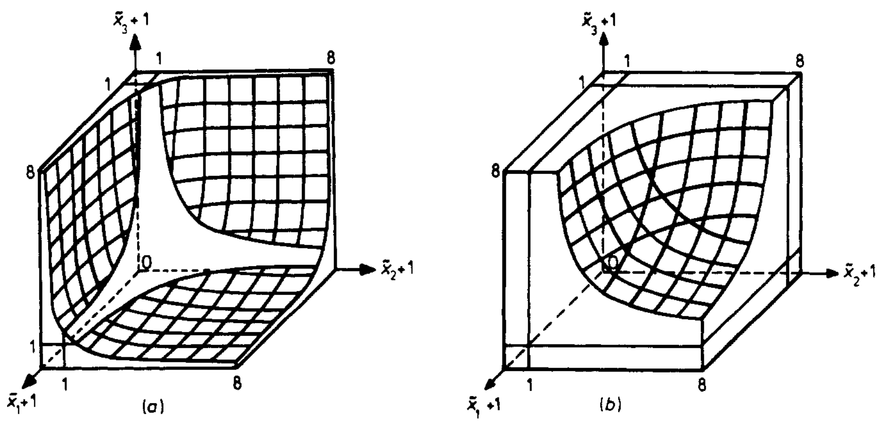

Figure 1. The fixed surface $D = s$ which coincides with the critical surface of paramagnetic–ferromagnetic transitions. $s = 3$.

(a) Triangular lattice: the cross-hatched portions of the surface are regions where one of the x_α (or K_α) is negative. (b) Honeycomb lattice: the surface lies entirely in the region where all of the $\tilde{x}_\alpha$ are positive.

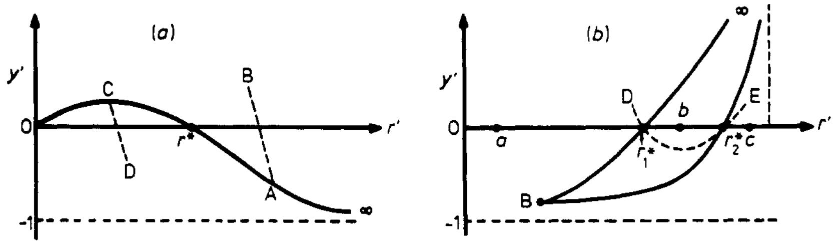

Figure 2. Typical mappings for a ray through the origin in x space.
(a) The ray intersects the fixed surface at a point with radial coordinate r^* and with all coupling constants positive. Critical lines not passing through $y'=0$, $r'=r^*$ such as AB and CD are ruled out by the uniqueness assumption. (b) The ray intersects the fixed surface at points with radial coordinates r_1^* and r_2^* and with one negative coupling constant. We suspect that the critical line is similar to DE.

$0<r<r^*$ is mapped onto the curve $r^*\mathrm{A}\infty$ by the transformation and the line segment $y=0$, $r>r^*$ onto the curve $\mathrm{OC}r^*$. Following Kim and Joseph we argue that if there is a unique critical line in the variables y', r', it must pass through the fixed point $y'=0$, $r'=r^*$. A critical line such as AB is ruled out by the uniqueness assumption since it would imply two critical points on the r' axis, point B and the point to the left of r^* which is mapped onto A by the transformation. Similar considerations rule out a critical line CD. Only critical lines with negative slope need be considered, since an increase in the three-spin coupling must be compensated by a decrease in tyhe two-spin couplings to remain on the critical line. Thus we have shown that for ferromagnetic couplings the fixed surface $D=s$, $y=0$ and the critical surface must coincide if a plausible uniqueness assumption is fulfilled.

The critical surface we have considered thus far is clearly associated with the paramagnetic–ferromagnetic transition since it passes through the region of ferromagnetic couplings $K_\alpha>0$. We now consider whether the fixed surface $D=s$ and the critical surface of paramagnetic–ferromagnetic transitions continue to coincide where all the couplings are not ferromagnetic. The cross-hatched portions of the fixed surface in figure 1(a) indicate regions where one of the three K_α is negative. Note that a ray through the origin in x-space which intersects the fixed surface at a point where one of the x_α is negative necessarily intersects the fixed surface twice. A typical mapping for such a ray is shown schematically in figure 2(b). r_1^* and r_2^* denote the fixed points at which the ray intersects the fixed surface $D=s$. The segment $y=0$, $a<r<b$ is mapped onto the upper curve $\mathrm{B}r_1^*\infty$ and the segment $y=0$, $b<r<c$ onto the lower curve $\mathrm{B}r_2^*\infty$. Other portions of the r axis are mapped onto the unphysical value of y' or r'. We are unable to construct as convincing an argument that the fixed surface and the critical surface must coincide as in the case considered above. If one assumes that the critical line in the variables y', r' intersects the r' axis twice (i.e. that the critical surface has roughly the same plausible topological features as the fixed surface $D=s$), certain types of critical lines which do not pass through the fixed points $y'=0$, $r'=r_1^*$ or r_2^* can be ruled out. We strongly suspect that the critical line is similar to DE in figure 2(b). However, we are unable to show that every critical line not passing through the fixed points violates the assumption of two critical points on the r' axis. Despite the difficulty of extending the arguments given above in the case of all $x_\alpha>0$, that the fixed surface and the critical surface of paramagnetic–ferromagnetic transitions continue to coincide in regions where one of the x_α is negative is almost certainly correct. It will be seen that this assumption leads to obvious criteria for the existence of a phase transition with one negative coupling and to exact results for the Ising model in the limit $s = 2$.

Expressed in terms of the K_α the critical surface for the triangular lattice defined by (6) has the form

$$e^{K_1+K_2+K_3} - (e^{K_1} + e^{K_2} + e^{K_3}) - (s-2) = 0. \tag{7}$$

The critical surface for the honeycomb lattice, obtained by combining (2) and (6), satisfies $D(s/\tilde{x}_1, s/\tilde{x}_2, s/\tilde{x}_3) = s$. This surface is shown in figure 1(b). In terms of the $\tilde{K}_\alpha$ the critical surface for the honeycomb lattice is given by

$$e^{\tilde{K}_1+\tilde{K}_2+\tilde{K}_3} - (e^{\tilde{K}_1+\tilde{K}_2} + e^{\tilde{K}_2+\tilde{K}_3} + e^{\tilde{K}_1+\tilde{K}_3}) - (s-1)(e^{\tilde{K}_1} + e^{\tilde{K}_2} + e^{\tilde{K}_3}) - (s^2 - 3s + 1) = 0. \tag{8}$$

Equations (7) and (8) reduce to known exact results in a number of special cases. For $K_1 = K_2 = K_3$ the results of Kim and Joseph for isotropic interactions are, of course, recovered. In the limit $K_3 = 0$ (or $\tilde{K}_3 = \infty$) an exact expression for the critical surface of the Potts model on a square lattice with different horizontal and vertical bonds, which follows readily from the self-duality property of the square lattice, is reproduced. The two-state Potts model is identical with the Ising model. For $s = 2$ equations (7) and (8) reduce to Houtappel's (1950) exact formulae for the critical surfaces of the anisotropic Ising model on the triangular and honeycomb lattices.

According to (7) there is a paramagnetic–ferromagnetic transition for the triangular lattice if and only if at least two of the coupling constants $J_\alpha = k_B T K_\alpha$ are positive and if in addition the J_α satisfy the inequalities $J_1 + J_2 > 0$, $J_2 + J_3 > 0$, $J_1 + J_3 > 0$. These inequalities guarantee the stability of the ferromagnetic ground state (Eggarter 1975, Tanaka and Uryû 1978).

For the honeycomb lattice stability of the ferromagnetic ground state requires that the inequalities $J_1 > 0$, $J_2 > 0$, $J_3 > 0$ be satisfied. Equation (8) predicts a transition for each set of J_α (see figure 1(b)) consistent with these inequalities. If one of the couplings goes to zero, the system decomposes into a set of one-dimensional chains, and the phase transition ceases to occur.

There are also surfaces in the region of all negative couplings which satisfy (7) and (8). Although we are unable to argue very convincingly on the basis of uniqueness assumptions that these surfaces must be critical surfaces, they reproduce exact results in the limit $s = 2$ when interpreted as such. The surface of negative couplings satisfying (7) only exists for $s \leq 2$. In the limit $s = 2$ it consists of the single point $K_1 = K_2 = K_3 = -\infty$. This limit is consistent with the exact results of Houtappel (1950), according to which the Ising antiferromagnet on the triangular lattice has no phase transition at finite temperatures. There are two surfaces of negative couplings which satisfy (8). One of these exists for $s \leq (3+\sqrt{5})/2 = 2{\cdot}62$ and the other for $s \leq (3-\sqrt{5})/2 = 0{\cdot}38$. For $s = 2$ the first of the surfaces agrees with exact results for the Ising model. In the case of the Ising model on the honeycomb lattice changing the sign of one or more of the J_α changes the ground state but not the critical temperature (Houtappel 1950). The surface of negative couplings which satisfies (8) for $s = 2$ is simply the reflection of the surface of positive couplings through the origin in K space. That neither of the solutions of (8) with negative couplings exists for integer values of s larger than 2 is also quite plausible, since the ground state of the system is then infinitely degenerate rather than two-fold degenerate as for $s = 2$.

We emphasise that we have not found all of the critical surfaces of the s-state Potts model for arbitrary signs of the coupling constants. For example, for $s = 3$ equation (7) describes the paramagnetic–ferromagnetic critical surface but not the critical surface of the paramagnetic–antiferromagnetic transition of the three-state Potts model. This transition has a four-component rather than a two-component order parameter and presumably belongs to a different universality class (Schick and Griffiths 1977). For $s=2$ equations (7) and (8) have paramagnetic–antiferromagnetic solutions with all couplings negative as well as paramagnetic–ferromagnetic solutions. However, neither (7) nor (8) have solutions for $s=2$ with two of the J_α negative, although it is clear that the critical temperature is the same as if the sign of the two negative J_α were reversed. For arbitrary signs of the coupling constants the combined duality and star–triangle transformations relate the free energies of the initial system and a system with transformed couplings which may be complex. However, the transformations only yield explicit expressions for the critical surface where there is a mapping across a fixed surface. The critical surfaces for $s=2$ with two negative coupling constants are examples of phase boundaries which are not fixed under the transformations considered here.

Note added in proof. Since this letter was accepted for publication, we have learned of a recent article (Baxter R J, Temperley H N V and Ashley S E 1978 *Proc. R. Soc.* A **358** 535–59) which contains similar results.

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